LECTURE 39 EXAMPLES FOR THE SUBSTITUTION METHOD

In this lecture, we go through a wide range of examples that use the method of substitution, in obvious and less obvious manners.

Example. (Trigonometry: rewrite before substitution)

(1) $\int \sin^2(x)\, dx$.
(2) $\int \cos^2(x)\, dx$.
(3) $\int \tan(x)\, dx$.
(4) $\int \sec(x)\, dx$.

Example. (Samplers of various derivative rules)

(1) $\int \frac{1}{x\ln(x)} dx$.
(2) $\int \frac{2}{1+25x^2} dx$.
(3) $\int \frac{dy}{\tan^{-1}\left(\frac{y}{7}\right)(49+y^2)}$.

Example. (Less obvious but direct substitution)

(1) $\int \frac{2z dz}{\sqrt[3]{z^2+1}}$.
(2) $\int \frac{1}{\sqrt{x}\left(2+3\sqrt{x}\right)^3} dx$.
(3) (rewrite after substitution) $\int \frac{x}{\sqrt{x-8}} dx$.
(4) (rewrite after substitution) $\int p\,(p+7)^8\, dp$.
(5) (rewrite after substitution) $\int x^3\sqrt{x^2+2}dx$.
(6) $\int \frac{1}{\sqrt{x}e^{-\sqrt{x}}} \csc^2\left(3e^{\sqrt{x}}-2\right) dx$.

Example. The velocity of a particle moving back and forth on a line is $v = \frac{ds}{dt} = 3\sin(6t)$ for all t. If $s = 0$ when $t = 0$, find the value of s when $t = \frac{\pi}{2}$ seconds.

Definite Integral Substitutions and the Area Between Curves

Recall that for indefinite integrals, we substitute in the following way.

$$\int f(g(x)) \cdot g'(x)\, dx \overset{u=g(x)}{=} \int f(u)\, du.$$

For definite integrals, all we need to be careful about, in addition to the above procedure, is the integrating limit. We are interesting in solving problems like the following,

$$\int_a^b f(g(x)) \cdot g'(x)\, dx.$$

Now, one makes a substitution $u = g(x)$. But the integrating limits are in x not in u, i.e. they read $x = a$ and $x = b$. So, to change to u expressions, we must convert them into $u = g(a)$ and $u = g(b)$. Thus,

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Example. Find $\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$.

Symmetries of the integrand greatly simplify its definite integral.

Theorem. *Let f be continuous on the symmetric integral $[-a, a]$.*

(1) *If f is even, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$.*
(2) *If f is odd, then $\int_{-a}^{a} f(x)\, dx = 0$.*

Remark. The interval must be symmetric about 0. To see why these claims are true, draw a picture.

Area Between Curves

Definition. If f and g are continuous with $f(x) \geq g(x)$ throughout $[a,b]$, then A, **area of the reigon between the curves** $y = f(x)$ **and** $y = g(x)$ **from** a **to** b, is the integral of $f - g$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Remark. You must identify which function is on the top and which one is on the bottom. Sketching the functions will help.

Sketching the functions will also increase your vocabulary of the types of scenarios two functions can create. For example, on $[a,b]$, two functions can intersect, or not intersect. If they don't intersect (mathematically means $f(x) = g(x)$ does not have a solution on $[a,b]$), then the area between them is the region below the top and above the bottom, and to the right of $x = a$ and to the left of $x = b$. If they do intersect, you must solve $f(x) = g(x)$ on $[a,b]$ to find the point(s) of intersection. If you find two intersections, then they become your limits of integration. If you only find one intersection, you need to identify the other boundary implicit in the problem, to decide the other limit of integration.

Example. Find the area of the region bounded above by the curve $y_1 = 2e^{-x} + x$ and below by the curve $y_2 = e^x/2$, on the left by $x = 0$ and on the right by $x = 1$.

Solution. Once you identified (see appendix for details, only if you are keen) that $y_1(x) \geq y_2(x)$ for all $x \in [0,1]$, then the problem is easy, you do

$$\begin{aligned}
A &= \int_0^1 (y_1(x) - y_2(x))\, dx \\
&= \int_0^1 \left(2e^{-x} + x - \frac{e^x}{2}\right) dx \\
&= \left[-2e^{-x} + \frac{x^2}{2} - \frac{e^x}{2}\right]_{x=0}^{x=1} \\
&= \left[\left(-2e^{-1} + \frac{1}{2} - \frac{e}{2}\right) - \left(-2 + 0 - \frac{1}{2}\right)\right] \\
&= -\frac{2}{e} + \frac{1}{2} - \frac{e}{2} - (-2) + \frac{1}{2} \\
&= 3 - \frac{2}{e} - \frac{e}{2}
\end{aligned}$$

Example. Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Another technique is to utilize the freedom of the coordinate against which you integrate. Suppose you are now given two curves $x = f(y)$ and $x = g(y)$, and are asked to find the area A between them over the interval $[c,d]$, then

$$A = \int_c^d [f(y) - g(y)]\, dy$$

where f is always the right-hand function and g is always the left-hand function.

Example. Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$. Do the problem in x and y coordinate respectively.

On the example with $2e^{-x} + x$ and $e^x/2$

Remark. This is example 4 in section 5.6. The book does NOT however justify the plots very well. It just shows the plot. Frankly, you have to be able to graph both functions to identify the region.

This is an unusual solution discussion. But I want to call to your attention to what I am really thinking. Here are the several questions I ask:

(1) In order to identify the top and bottom function, you need to prove either $y_1(x) \geq y_2(x)$ or $y_2(x) \geq y_1(x)$ for $x \in [0,1]$. But, it is very hard to do in this problem. You are comparing

$$y_1 = 2e^{-x} + x, \quad y_2 = e^x/2$$

and ask which one is **always** bigger on $[0,1]$. Not an obvious job.

(2) Another strategy is to check whether they have intersections on this interval. If they don't, it means on this interval they don't cross each other at all, and hence are ordered. But, checking intersections means checking if there is a solution to $y_1 = y_2$, or $2e^{-x} + x = \frac{e^x}{2}$. Is this simple to solve without looking it up? No.

(3) However, it is reasonable to check the following.

We know how the left endpoints look like.

$$y_1(0) = 2 > \frac{1}{2} = y_2(0)$$

so y_1 starts on top of y_2. Now, it would make sense to check if the minimum of y_1 is bigger than the maximum of y_2 on $[0,1]$, because if it is, then y_1 and y_2 never touch. We find that

$$y_1'(x) = -2e^{-x} + 1$$

which means it has a critical point at

$$e^{-x} = \frac{1}{2} \implies x = \ln(2).$$

We find $y_1(0) = 2$, $y_1(1) = \frac{2}{e} + 1$ and $y_1(\ln(2)) = 2e^{-(\ln 2)} + x = 1 + \ln(2)$. So, which one is the smallest? The calculator tells me $1 + \ln(2)$ is the smallest, only by a small margin. Thus, the minimum of $y_1(x)$ on $[0,1]$ is $1 + \ln(2)$. The maximum of y_2 is the right endpoint because it is a monotone increasing function,

$$\max y_2(x) = \frac{e}{2} \approx 1.359$$

which is less than $1 + \ln(2) \approx 1.693$. Hence, the two functions don't touch, and remained ordered on $[0,1]$ that $y_1(x) \geq y_2(x)$. We are done, with the help of a calculator.

(4) You may wonder how to show part 3 without using a calculator. And I am gonna tell you that it is hard!

Among the endpoints and critical point, we find definitely that $y_1(0) = 2$ is the largest, of course. So, we compare the other two values $\frac{2}{e} + 1$ and $\ln(2) + 1$, or simply $\frac{2}{e}$ and $\ln(2)$. How do you compare two numbers? We know if $e^x > e^y$ then $x > y$ since the exponential function is monotone increasing. Thus, we now compare $e^{\frac{2}{e}}$ and $e^{\ln(2)} = 2$. We raise both sides to the e^{th} power to now compare e^2 and 2^e. And now the struggle is real.